Equivariant Seidel maps and a flat connection on equivariant symplectic cohomology

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An intertwining relation for equivariant Seidel maps
arXiv:2010.03342

Shift operators and connections on equivariant symplectic cohomology
arXiv:2104.01891

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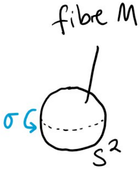
Seidel maps

Setup and definition

(M, ω) is a closed symplectic manifold.
$\sigma : S^1 \times M \to M$ is a Hamiltonian S^1-action.

The **clutching bundle** $E(\sigma)$ is a bundle with fibre M over the sphere S^2.



The **quantum Seidel map**

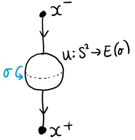
$$QS(\sigma) : QH\ \ (M) \to QH\ \ ^{+|\sigma|}(M)$$

counts pseudoholomorphic sections of the clutching bundle $E(\sigma) \to S^2$.

Novikov ring is $\Lambda = Z[q^{H_2(M)}]$ $\quad q^A$ with $A \quad H_2(M)$.
Quantum cohomology is $QH\ \ (M) = H\ \ (M) \quad \Lambda$.

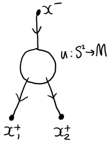
Novikov ring is $\Lambda = \mathbb{Z}[q^{H_2(M)}]$ q^A with A $H_2(M)$.
Quantum cohomology is $QH\ (M) = H\ (M)$ Λ.

$$QS(\sigma)(x^+) = \sum_{\substack{A\ \ H_2(M) \\ x^-\ \ H\ (M)}} \#\Big(\quad \Big) q^A x^-$$



Quantum product is

$$x_1^+ \quad x_2^+ = \sum_{\substack{A\ \ H_2(M) \\ x^-\ \ H\ (M)}} \#\Big(\quad \Big) q^A x^-$$

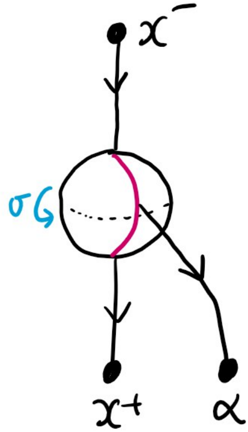


Theorem (Seidel, '97)

We have $QS(\sigma)(\alpha \quad x^+) = \alpha \quad QS(\sigma)(x^+)$ for α $QH\ (M)$.

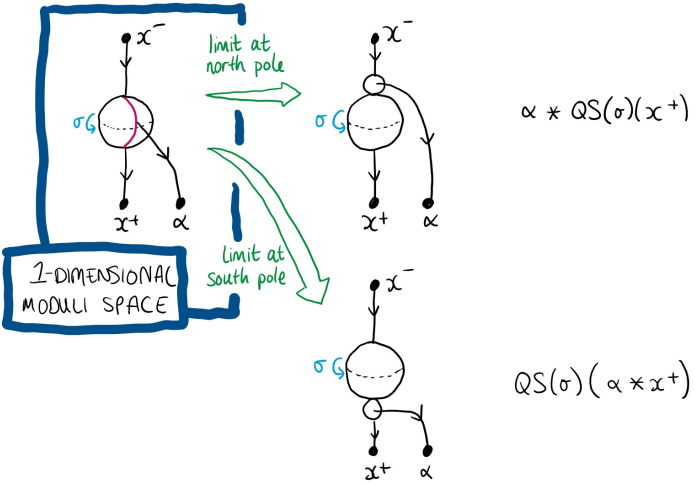
Seidel maps

Proof of $QS(\sigma)(\alpha \quad x^{+}) = \alpha \quad QS(\sigma)(x^{+})$



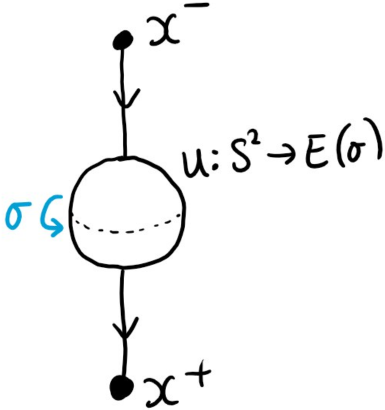
Seidel maps

Proof of $QS(\sigma)(\alpha \quad x^{+}) = \alpha \quad QS(\sigma)(x^{+})$



We showed $QS(\sigma)(\alpha \quad x^{+}) = \alpha \quad QS(\sigma)(x^{+})$.
Therefore, we have $QS(\sigma)(x) = x \quad QS(\sigma)(1)$.
This yields the **Seidel representation**

$$\begin{aligned} \left\{ \text{Hamiltonian } S^1\text{-actions} \right\} &\rightarrow QH \;\; (M)^{\times} \\ \sigma &\rightarrow QS(\sigma)(1). \end{aligned} \tag{1}$$



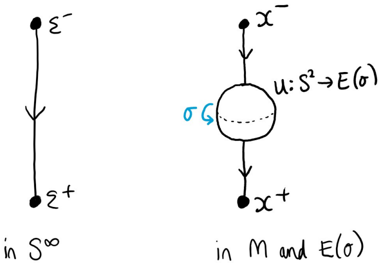
x^-
$u: S^2 \to E(\sigma)$
σ
x^+

Equivariant Seidel maps

Definition

In the Borel construction,
we have a fibre bundle

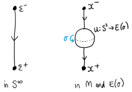
$$X \longrightarrow S^{\infty} \times_{S^1} X$$
$$\downarrow$$
$$\frac{S^{\infty}}{S^1} = \mathrm{CP}^{\infty}$$



S^1**-equivariant quantum Seidel map** is

$$QS_{S^1}(\sigma) : QH_{S^1}(M, \sigma) \to QH_{S^1}^{+|\sigma|}(M, \mathrm{Id})$$

$$QS_{S^1}(\sigma)(\varepsilon^+, x^+) = \sum_{\substack{A \quad H_2(M) \\ (\varepsilon^-, x^-) \quad H_{S^1}(M, \mathrm{Id})}} \#(\quad)q^A(\varepsilon^-, x^-)$$



Theorem (Intertwining relation, TL-J)

We have

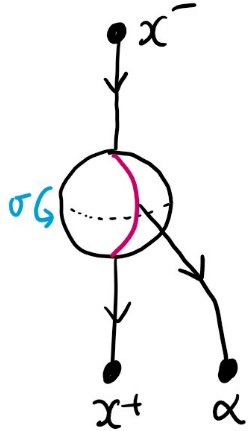
$$QS_{S^1}(\sigma)(x \quad \alpha^+) - QS_{S^1}(\sigma)(x) \quad \alpha^- = \mathbf{u}\, WQS_{S^1}(\sigma,\alpha)(x) \qquad (2)$$

for any class $\alpha \quad H^2_{S^1}(E(\sigma))$.
Here, $\alpha^\pm$ *are the restrictions of* α *to the fibres above the poles,* $\mathbf{u} \quad H^2(\mathrm{CP}^\infty)$ *is the generator of* $H \ (\mathrm{CP}^\infty)$*, and* WQS_{S^1} *is a weighted version of the* S^1*-equivariant quantum Seidel map.*

We proved this for all $\alpha \quad H_{S^1}(E(\sigma))$ but we'll only present $|\alpha| = 2$ for simplicity.

Equivariant Seidel maps

Proof of $QS_{S^1}(\sigma)(x \quad \alpha^{+}) - QS_{S^1}(\sigma)(x) \quad \alpha^{-} = \mathbf{u}\, WQS_{S^1}(\sigma, \alpha)(x)$



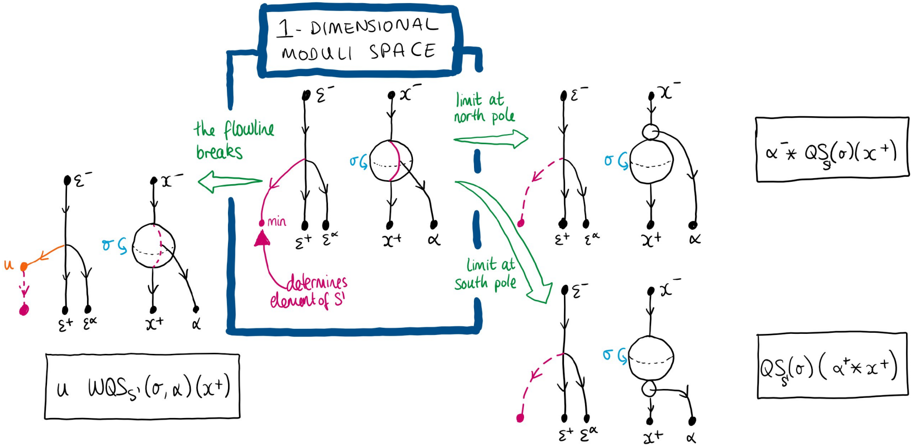
To parameterise the line of longitude, we would need an S^1-equivariant map $S^\infty \to S^1$.

But none exists.
Let $w \quad S^\infty$. The composition $S^1 \cdot w \hookrightarrow S^\infty \to S^1$ is an isomorphism.
But S^∞ is contractible, so $\pi_1(S^\infty) = 0$.

Key insight: it is sufficient to define $S^\infty \to S^1$ on a *generic* subset $W \quad S^\infty$.

Equivariant Seidel maps

Proof of $QS_{S^1}(\sigma)(x \quad \alpha^+) - QS_{S^1}(\sigma)(x) \quad \alpha^- = \mathbf{u}\ WQS_{S^1}(\sigma, \alpha)(x)$



Equivariant Seidel maps

Algebraic geometry

Maulik and Okounkov defined equivariant Seidel maps in 2013.
They also proved the intertwining relation for $\alpha \quad H^2_{S^1}(E(\sigma))$.
Iritani gave a similar construction in a different setting in 2017.

Their definitions use *virtual fundamental classes* to count sections.
Their proofs of the intertwining relation use *virtual localisation*.

We are interested in S^1-equivariant Floer theory, which does not have the above machinery.
We redefined $QS_{S^1}(\sigma)$ using a Morse Borel construction.
We reproved the intertwining relation with a new Morse homotopy proof using a 1-dimensional moduli space argument.

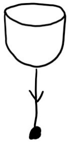
The **Floer Seidel map**

$$FS(\sigma) : FH\ (M;H) \to FH^{\ +|\sigma|}(M;\sigma\ H)$$

maps the Hamiltonian orbit $x : S^1 \to M$ to the orbit

$$(\sigma\ x)(t) = \sigma_t^{-1}(x(t)), \qquad t \quad S^1.$$

$FS(\sigma)$ is an isomorphism of cochain complexes.



Compact: $QH\ (M) = FH\ (M;H)$ for all Hamiltonians H.
The Floer Seidel map and the quantum Seidel map agree.

Non-compact: (with convexity assumption, for example $O_{\mathbb{P}^1}(-1)$)
$FH\ (M,\lambda;H)$ depends on the slope λ ($H = \lambda R +$ const. at infinity)
Symplectic cohomology is $SH\ (M) = \varinjlim FH\ (M,\lambda)$ as $\lambda \to \infty$.
For *linear* σ of slope κ, the Floer Seidel map is

$$FS(\sigma) : FH\ (M,\lambda;H) \to FH^{\ +|\sigma|}(M,\lambda-\kappa;\sigma\ H).$$

The quantum Seidel map is only defined for linear σ with $\kappa \geq 0$.

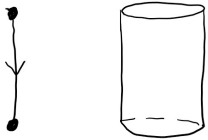
The loop space $LM = \{x : S^1 \to M\}$ has an S^1-action given by

$$(\theta \cdot x)(t) = x(t - \theta) \qquad \theta \quad S^1. \tag{3}$$

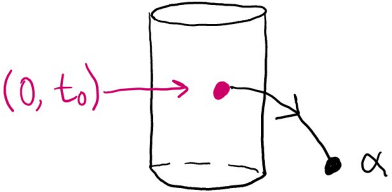
Definition (Equivariant Floer cohomology)

We combine Morse theory on S^∞ with Floer theory on M.
The Hamiltonian $H : S^\infty \times S^1 \times M \to \mathbb{R}$ now depends on S^∞.
The **equivariant Floer cochain complex** is generated over Λ by $[\varepsilon, x]$, where $\varepsilon \quad S^\infty$ is critical and x is a Hamiltonian orbit of H_ε.
The differential counts flowlines in S^∞ paired with Floer cylinders in M.

$FH_{S^1}(M, \lambda; H)$ is a $\Lambda \quad \mathbb{Z}[\mathbf{u}]$-module.



There is a map $QH\ \ (M)\quad FH\ \ (M, \lambda) \to FH\ \ (M, \lambda)$ which counts



An equivariant version of this map would use a map $S^{\infty} \to S^1$ defined on a *generic* subset $W \quad S^{\infty}$.
Therefore it would not be a chain map.

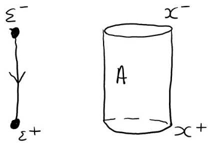
Equivariant Floer cohomology

Differentiation

Novikov ring is $\Lambda = Z[q^{H_2(M)}] \quad q^A$ with $A \quad H_2(M)$.
For $\alpha \quad H^2(M)$, define

$$\frac{d}{d\alpha}(q^A) = \alpha(A)\, q^A. \tag{4}$$

We can pick a Λ-basis for the equivariant Floer cochain complex and apply $\frac{d}{d\alpha}$. *This is not a chain map either.*



$$\frac{d}{d\alpha}[\varepsilon^+, x^+] = 0 \qquad \partial \frac{d}{d\alpha}[\varepsilon^+, x^+] = 0$$

$$\partial[\varepsilon^+, x^+] = q^A[\varepsilon^-, x^-] \qquad \frac{d}{d\alpha}\partial[\varepsilon^+, x^+] = \alpha(A) q^A [\varepsilon^-, x^-]$$

On $QH_{S^1}(M, \mathrm{Id}) = \Lambda \quad \mathbb{Z}[\mathbf{u}] \quad H\ (M)$, the **Dubrovin connection** is

$$ {}_{\alpha}(q^A x) = \mathbf{u}\frac{d}{d\alpha}(q^A)x + \alpha \quad q^A x. \tag{5}$$

Theorem (Connection, TL-J)

On $FH_{S^1}(M, \lambda; H)$, for $\alpha \quad H^2(M)$ the map

$$ {}_{\alpha} = \mathbf{u}\frac{d}{d\alpha} + (\alpha \quad \cdot) - w_{\alpha} \tag{6}$$

is a chain map on the equivariant Floer cochain complex.

Seidel proved special case $\alpha = [\omega]$ in 2016.

Theorem (Flatness, TL-J)

On $FH_{S^1}(M, \lambda; H)$, for any $\alpha, \beta \quad H^2(M)$, we have

$${}_{\alpha} \circ \quad {}_{\beta} = \quad {}_{\beta} \circ \quad {}_{\alpha}.$$

The Dubrovin connection is $\quad {}_{\alpha} = \mathbf{u}\frac{d}{d\alpha} + \alpha \quad$.
A calculation shows the intertwining relation

$$QS_{S^1}(\sigma)(x \quad \alpha^{+}) - QS_{S^1}(\sigma)(x) \quad \alpha^{-} = \mathbf{u}\; WQS_{S^1}(\sigma, \alpha)(x)$$

is equivalent to

$$QS_{S^1}(\sigma)(\quad {}_{\alpha}(x)) - \quad {}_{\alpha}(QS_{S^1}(\sigma)(x)) = 0.$$

Theorem (Connection and Floer Seidel map, TL-J)

On $FH_{S^1}(M, \lambda; H)$, for any $\alpha \quad H^2(M)$ and any linear σ, we have

$${}_{\alpha} \circ FS_{S^1}(\sigma) = FS_{S^1}(\sigma) \circ \quad {}_{\alpha}.$$

Now let T be a torus acting on M.
S^1-actions correspond to **cocharacters** of T, which are group maps $\sigma : S^1 \to T$.
Let $\widehat{T} = S^1_0 \times T$.
Constructions of $QH_{\widehat{T}}(M)$, $FH_{\widehat{T}}(M, \lambda)$, $SH_{\widehat{T}}(M)$.
We get $\ _{\alpha}$, $QS_{\widehat{T}}(\sigma)$, $FS_{\widehat{T}}(\sigma)$ too.
But now we can undo the change of $\widehat{T}$-action, to get endomorphisms

$$\begin{aligned} \mathrm{S}_\sigma &: QH_{\widehat{T}}(M) \to QH_{\widehat{T}}^{+|\sigma|}(M) \\ \mathrm{S}_\sigma &: SH_{\widehat{T}}(M) \to SH_{\widehat{T}}^{+|\sigma|}(M) \end{aligned} \tag{7}$$

called **shift operators**.

Shift operators

Example: $\mathbb{P}^2$

The torus T^2 acts on P^2 (on middle and last coordinate).
We have $H\ \ (BT) = \mathrm{Z}[t_1, t_2]$ and $\Lambda = \mathrm{Z}[q]$ with $|q| = 6$.

$$QH_{\widehat{T}}(\mathrm{P}^2) = \frac{\Lambda \quad \mathrm{Z}[x_0, x_1, x_2, \mathbf{u}]}{x_0 x_1 x_2 - q} \tag{8}$$

We calculate $\ \ _x = \mathbf{u}(q\frac{d}{dq}) + x_0$.
Let σ correspond to rotation of middle coordinate. We have:

$$\begin{aligned}
\mathrm{S}_\sigma(1) &= x_1 & \mathrm{S}_\sigma(t_1 y) &= (t_1 + \mathbf{u})\mathrm{S}_\sigma(y) \\
\mathrm{S}_\sigma(x_0) &= x_1 x_0 & \mathrm{S}_\sigma(t_2 y) &= t_2 \mathrm{S}_\sigma(y) \\
\mathrm{S}_\sigma(x_1) &= x_1(x_1 - \mathbf{u}) & & \\
\mathrm{S}_\sigma(x_2) &= x_1 x_2 & &
\end{aligned} \tag{9}$$

We had the **Seidel representation**

$$\begin{aligned}\left\{\text{Hamiltonian } S^1\text{-actions}\right\} &\to QH\ \ (M)^\times \\ \sigma &\to QS(\sigma)(1).\end{aligned} \tag{10}$$

We also have $\mathsf{S}_{\sigma}\mathsf{S}_{\sigma}\ = \mathsf{S}_{\sigma+\sigma}$.
This yields

$$\mathsf{S} : \mathrm{Cochar}^{+}(T) \to \mathrm{End}_{\Lambda[\mathbf{u}]}(QH_{\widehat{T}}(M))$$

$$\mathsf{S} : \mathrm{Cochar}(T) \to \mathrm{Aut}_{\Lambda[\mathbf{u}]}(SH_{\widehat{T}}(M)).$$

Conclusion

We have expanded the algebraic structures on $SH_{\widehat{T}}(M)$ with

- a flat connection $_{\alpha}$,
- shift operators S_{σ}.

They are compatible, computable in examples and capture geometric information.

Thanks for your attention.

Definition (Borel construction of S^1-equivariant cohomology)

X is topological space, ρ is S^1-action on X.
Take S^∞: it's a contractible space with a free S^1-action.

Borel quotient is $S^\infty \times X/$, where is $(\theta \cdot w, x)$ $(w, \rho_\theta(x))$.
It's denoted $S^\infty \times_{S^1} X$.

S^1-equivariant cohomology is $H_{S^1}(X, \rho) = H\ (S^\infty \times_{S^1} X)$.

The projection map $S^\infty \times_{S^1} X \to S^\infty / S^1 = \mathrm{CP}^\infty$ induces a map $H\ (\mathrm{CP}^\infty) = \mathrm{Z}[\mathbf{u}] \to H_{S^1}(X, \rho)$.

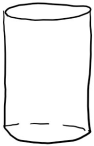
Extra information

Floer cohomology definition

The **Floer cohomology** $FH\ (M)$ is inspired by Morse cohomology on the loop space $LM = \{x : S^1 \to M\}$.

Take a function $H : S^1 \times M \to \mathbb{R}$, called a **Hamiltonian function**. Define the **Hamiltonian vector field** by $\omega(\cdot, X_t) = \mathrm{d}H_t$. The **Hamiltonian orbits** are the curves $x : S^1 \to M$ that follow X_t. The **Floer cochain complex** is freely generated over Λ by the Hamiltonian orbits.

A **Floer cylinder** is a cylinder $u : \mathbb{R} \times S^1 \to M$ which satisfies a **Floer equation**. The **Floer differential** counts Floer cylinders between the Hamiltonian orbits.



Definition (Convex symplectic manifold)

A **convex symplectic manifold** is the union of a compact symplectic manifold and the symplectic manifold $([1, \infty) \times \Sigma, \mathrm{d}(R\alpha))$, where (Σ, α) is a closed contact manifold.

Example $O_{\mathbb{P}^1}(-1)$, where $\Sigma = S^3$ is the sphere bundle.

Floer cohomology depends on the **slope** λ, where $H = \lambda R +$ constant at infinity.

Symplectic cohomology $SH^*(M)$ is the limit of $FH^*(M, \lambda)$ as $\lambda \to \infty$.